Magnetohydrodynamic wakes

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The most notable feature of the magnetohydrodynamic flow at large distances from a three-dimensional body is the formation of two wakes, within which vorticity and electric current are confined. In this paper results are obtained for the effective diffusivity and the relation between current and vorticity in each wake, for the balance between the strengths of the disturbances in the wakes and in the irrotational current-free flow outside, and for the lift and drag forces acting on the body. The final answers take the form of remarkably simple extensions of the corresponding formulae for non-conducting flow. In spite of the extra wake and the presence of a magnetic as well as a velocity field, the flow perturbation at large distances still has only three degrees of freedom.

1. Introduction

For the flow of a viscous conducting fluid past a body in a magnetic field, it has been recognized for some time that there are two wakes, in general, instead of one as for a non-conducting fluid. Vorticity and electric current are zero outside the wakes, which lie in the Alfvén directions, i.e. those directions in which are to be found disturbances which originate at the body and move, relative to the fluid, along the magnetic field lines with the Alfvén speed. The wakes are indeed often referred to as Alfvén waves, particularly when the effects of diffusion are being ignored. This paper is devoted to a detailed study of these wakes, for steady three-dimensional magnetohydrodynamic flow.

As in the equivalent non-conducting problem, the solutions obtained are valid only at sufficiently large distances, greater than r_0 say, from the body. The distance r_0 must be such that it is legitimate to neglect squares of disturbances to the stream values of the quantities concerned, and also such that the Reynolds number and the magnetic Reynolds number based on r_0 shall be large. (It is not necessary that their values based on a typical body dimension shall be large.) Since both viscous and magnetic diffusivity are included in the analysis there can be no doubt that the disturbances are small at large distances, even in the wake regions.

The whole analysis is an extension of that given by Hasimoto (1960) for two-dimensional flow. Hasimoto only claimed that his results applied in three special cases: (1) with the magnetic and velocity fields alined at large distances; (2) with the magnetic and viscous diffusivities equal; and (3) for Stokes flow. Case (1) is also a special case in our analysis here. However, our main concern is with the general case, with the fields not alined, and here we show that the restriction (2) is unnecessary. As for (3), the Stokes approximation ceases to be valid at large distances whatever the Reynolds number (as for non-conducting flow), and we shall not discuss this case here.

An interesting fact which emerges during the investigation is that for undisturbed velocity and magnetic fields inclined at an angle α, the details of the flow in the wakes in the limit as $\alpha \to 0$ is not the same as for alined fields, with $\alpha = 0$. A physical explanation is that for $\alpha \neq 0$ the two wakes develop independently, while for $\alpha = 0$ Alfvén disturbances can propagate from a point in a wake with either of the two Alfvén velocities and still remain within the wake. This suggests that caution is needed when making deductions from solutions for alined fields in respect to what occurs when the fields are not alined.

This same point has arisen in analyses which assume the fluid to be inviscid and perfectly conducting. Thus for the flow past a thin aerofoil, Stewartson (1961) has shown that the results of Sears & Resler (1959) for $\alpha = 0$ are not approached in the limit as $\alpha \to 0$. In addition, Stewartson (1960) has argued that Sears & Resler's solution is not unique, on the grounds that they are not justified in assuming that the flow perturbations are negligible at large distances. The results of the present paper may help to resolve such questions, since disturbances in wakes must be represented by equivalent singularities in the corresponding perfectly conducting, inviscid flow.

The effects of finite conductivity on the two-dimensional inviscid flow past a thin aerofoil with the magnetic field at right angles to the stream, the second configuration examined by Sears & Resler (1959), were studied briefly by them and in greater detail by McCune (1960). Arguing by analogy with the flow past a wavy wall, McCune deduced how diffusion modifies the pressure distribution over the aerofoil for large magnetic Reynolds numbers, and also how the current and vorticity are damped with increasing distance. McCune refers to the 'depth of penetration' of the current density, which is perhaps somewhat misleading, since his solution for the current in each wake has a doublet strength which is not attenuated, in agreement with the prediction of this paper. It is true that diffusion causes the current density to fall off indefinitely at large distances along the wake, but the flow and field perturbations outside the wake are not affected by the diffusion. Lary (1962) has investigated the same problem for an alined magnetic field, and obtained results which are applicable for sufficiently small values of the magnetic Reynolds number, such that the thickness of the magnetic boundary layer is large compared with the aerofoil's thickness or centre-line displacement.

2. Structure of the wakes

The equations governing the steady three-dimensional magnetohydrodynamic flow of a conducting fluid of constant properties are

$$(\mathbf{q}.\nabla)\,\mathbf{q} = -(1/\rho)\,\nabla p + \nu\nabla^2\mathbf{q} + (\mu/\rho)\,\mathbf{j}\times\mathbf{H}, \tag{2.1}$$

$$\mathbf{j} = \nabla\times\mathbf{H} = \sigma(\mathbf{E}+\mu\mathbf{q}\times\mathbf{H}), \tag{2.2}$$

$$\nabla.\mathbf{q} = \nabla.\mathbf{H} = \nabla\times\mathbf{E} = 0, \tag{2.3}$$

in M.K.S. units, where p is the pressure, $\mathbf{q}$ the velocity, ρ the density, ν the kinematic viscosity, μ the permeability, σ the conductivity, $\mathbf{H}$ the magnetic field, $\mathbf{E}$ the electric field and $\mathbf{j}$ the current. Taking the curl of (2.2) and using (2.3) we obtain

$$(\mathbf{q}\,.\nabla)\,\mathbf{H}-(\mathbf{H}\,.\nabla)\,\mathbf{q} = \eta\nabla^2\mathbf{H}, \tag{2.4}$$

where $\eta = (\sigma\mu)^{-1}$ is the magnetic diffusivity. Also (2.1) can be written as

$$(\mathbf{q}\,.\nabla)\,\mathbf{q}-(\mu/\rho)\,(\mathbf{H}\,.\nabla)\,\mathbf{H} = -(1/\rho)\,\nabla(p+\tfrac{1}{2}\mu H^2)+\nu\nabla^2\mathbf{q}. \tag{2.5}$$

Let the undisturbed velocity and magnetic fields be $\mathbf{U}_0 = U_0\mathbf{a}$ and $\mathbf{H}_0 = H_0\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are unit vectors inclined at an angle α. (We may assume without loss of generality that $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$, since if $\mathbf{H}$ is replaced by $-\mathbf{H}$ the whole flow is unaltered.) At large distances the perturbations due to a finite body must be small, even in wakes, so we write

$$\mathbf{q} = U_0\mathbf{a}+\mathbf{v}, \quad \mathbf{H} = H_0\mathbf{b}+\mathbf{h}, \tag{2.6}$$

and neglect squares and products of $\mathbf{v}$ and $\mathbf{h}$. Equations (2.4) and (2.5) become

$$U_0(\mathbf{a}\,.\nabla)\,\mathbf{h}-H_0(\mathbf{b}\,.\nabla)\,\mathbf{v} = \eta\nabla^2\mathbf{h}, \tag{2.7}$$

$$U_0(\mathbf{a}\,.\nabla)\,\mathbf{v}-(\mu H_0/\rho)\,(\mathbf{b}\,.\nabla)\,\mathbf{h} = -(1/\rho)\,\nabla(p+\tfrac{1}{2}\mu H^2)+\nu\nabla^2\mathbf{v}. \tag{2.8}$$

Now take the curl of these equations. Since the vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{v}$ and the current $\mathbf{j} = \nabla\times\mathbf{h}$, we obtain

$$U_0(\mathbf{a}\,.\nabla)\,\mathbf{j}-H_0(\mathbf{b}\,.\nabla)\,\boldsymbol{\omega} = \eta\nabla^2\mathbf{j}, \tag{2.9}$$

$$U_0(\mathbf{a}\,.\nabla)\,\boldsymbol{\omega}-(\mu H_0/\rho)\,(\mathbf{b}\,.\nabla)\,\mathbf{j} = \nu\nabla^2\boldsymbol{\omega}. \tag{2.10}$$

Eliminating $\mathbf{j}$ between (2.9) and (2.10) gives

$$\eta\nu\nabla^4\boldsymbol{\omega}+(\eta+\nu)\,U_0(\mathbf{a}\,.\nabla)\,\nabla^2\boldsymbol{\omega}+U_0^2\{(\mathbf{a}+\beta^{\frac{1}{2}}\mathbf{b})\,.\nabla\}\{(\mathbf{a}-\beta^{\frac{1}{2}}\mathbf{b})\,.\nabla\}\,\boldsymbol{\omega} = 0, \tag{2.11}$$

where $\beta = \mu H_0^2/\rho U_0^2$ is the square of the ratio of the Alfvén speed to the fluid speed in the undisturbed flow. This is the basic equation for the vorticity $\boldsymbol{\omega}$ at large distances. Exactly the same equation is obeyed by the current $\mathbf{j}$.

At very large distances all gradients are small and the lowest-order derivatives must predominate. These occur in the last term of (2.11), and hence the main disturbances are to be found only in the directions given by the unit vectors $\mathbf{s}_1$, $\mathbf{s}_2$, where

$$\mathbf{s}_1 = (1+2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{-\frac{1}{2}}\,(\mathbf{a}+\beta^{\frac{1}{2}}\mathbf{b}), \quad \mathbf{s}_2 = (1-2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{-\frac{1}{2}}\,(\mathbf{a}-\beta^{\frac{1}{2}}\mathbf{b}), \tag{2.12}$$

as shown in figure 1. There are thus two wakes, in these two directions. The wakes occur in the tracks of disturbances which leave the body and travel, relative to the fluid, at the Alfvén speed $(\mu/\rho)^{\frac{1}{2}}\,H_0$ parallel to the magnetic field $\mathbf{H}_0$, in either direction. We shall use suffixes 1 and 2 to denote quantities associated with the wakes in the directions $\mathbf{s}_1$ and $\mathbf{s}_2$ respectively.

To study the vorticity $\boldsymbol{\omega}_1$ in the $\mathbf{s}_1$ wake take rectangular co-ordinates (s_1, t_1, n_1), where the unit vector $\mathbf{n}_1$ is perpendicular to $\mathbf{s}_1$ in the plane of $\mathbf{a}$ and $\mathbf{b}$, as shown in figure 1, and $\mathbf{t}_1$ is in the direction of $\mathbf{b}\times\mathbf{a}$. Thus

$$\left.\begin{aligned}\mathbf{t}_1 &= \operatorname{cosec}\alpha\,\mathbf{b}\times\mathbf{a},\\ \mathbf{n}_1 &= \operatorname{cosec}\alpha(1+2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{-\frac{1}{2}}\{-(\cos\alpha+\beta^{\frac{1}{2}})\,\mathbf{a}+(1+\beta^{\frac{1}{2}}\cos\alpha)\,\mathbf{b}\}.\end{aligned}\right\} \tag{2.13}$$

Note that $\mathbf{t}_1 = \mathbf{t}_2 = \mathbf{t}$, say. Equation (2.11) becomes

$$\eta\nu\nabla^4\boldsymbol{\omega}_1 - (\eta+\nu)\,U_0(1+2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{-\frac{1}{2}}\left\{(1+\beta^{\frac{1}{2}}\cos\alpha)\frac{\partial}{\partial s_1} - \beta^{\frac{1}{2}}\sin\alpha\frac{\partial}{\partial n_1}\right\}\nabla^2\boldsymbol{\omega}_1 + U_0^2\frac{\partial}{\partial s_1}\left\{(1-\beta)\frac{\partial}{\partial s_1} - 2\beta^{\frac{1}{2}}\sin\alpha\frac{\partial}{\partial n_1}\right\}\boldsymbol{\omega}_1 = 0, \quad (2.14)$$

where now $\nabla^2 = \partial^2/\partial s_1^2 + \partial^2/\partial t_1^2 + \partial^2/\partial n_1^2$. All derivatives are small in the wake, and from general wake theory we may assume that

$$\frac{\partial}{\partial s_1} \ll \frac{\partial}{\partial n_1}.$$

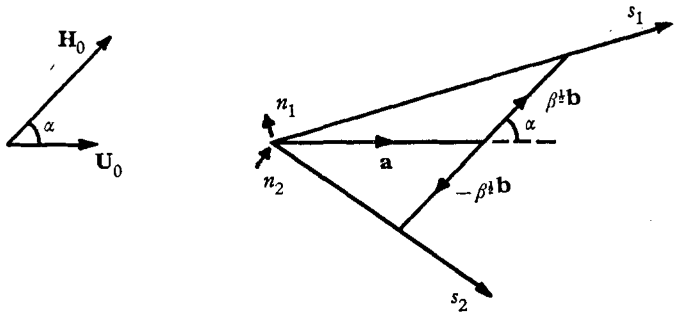

FIGURE 1. The location of the wakes and the co-ordinate systems used.

If α is not small, the major terms of (2.14) give

$$(\eta+\nu)\,(1+2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{-\frac{1}{2}}\frac{\partial}{\partial n_1}\nabla^2\boldsymbol{\omega}_1 = 2U_0\frac{\partial^2\boldsymbol{\omega}_1}{\partial s_1\,\partial n_1},$$

and hence

$$\nabla^2\boldsymbol{\omega}_1 = \frac{2U_0}{\eta+\nu}(1+2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{\frac{1}{2}}\frac{\partial\boldsymbol{\omega}_1}{\partial s_1}, \quad (2.15)$$

since the vorticity is zero outside the wake regions. Here ∇^2 may be approximated to by $\partial^2/\partial t_1^2 + \partial^2/\partial n_1^2$. We note that

$$\frac{\partial}{\partial s_1} = O\left(\frac{\partial}{\partial n_1}\right)^2,$$

which is in accord with the assumption made above. For the $\mathbf{s}_2$ wake all we have to do is to replace $\beta^{\frac{1}{2}}$ by $-\beta^{\frac{1}{2}}$. The equation corresponding to (2.15) is therefore

$$\nabla^2\boldsymbol{\omega}_2 = \frac{2U_0}{\eta+\nu}(1-2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{\frac{1}{2}}\frac{\partial\boldsymbol{\omega}_2}{\partial s_2}. \quad (2.16)$$

These equations are of precisely the same form as that governing the wake in a non-conducting fluid, namely $\nabla^2\boldsymbol{\omega} = (U_0/\nu)\,\partial\boldsymbol{\omega}/\partial x$. The spread of vorticity in the two wakes is therefore exactly as in a non-conducting fluid of kinematic viscosity

$$\tfrac{1}{2}(\eta+\nu)\,(1\pm 2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{-\frac{1}{2}}. \quad (2.17)$$

The wakes from a point disturbance are paraboloidal in shape, with widths proportional to

$$(\eta+\nu)^{\frac{1}{2}}\,U_0^{-\frac{1}{2}}(1\pm 2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{-\frac{1}{4}}\,s_{1,2}^{\frac{1}{2}}.$$

The relationship between the vorticity and current fields in the wakes is given by (2.9) or (2.10). Thus for the $\mathbf{s}_1$ wake the major terms in each equation are those involving $\partial/\partial n_1$ on the left-hand side. Each equation leads to the results

$$\mathbf{j}_1 = -(\rho/\mu)^{\frac{1}{2}}\,\boldsymbol{\omega}_1, \quad \mathbf{j}_2 = (\rho/\mu)^{\frac{1}{2}}\,\boldsymbol{\omega}_2. \tag{2.18}$$

Equivalent results for the corresponding velocity and magnetic fields can be deduced from (2.7), and are

$$\mathbf{h}_1 = -(\rho/\mu)^{\frac{1}{2}}\,\mathbf{v}_1, \quad \mathbf{h}_2 = (\rho/\mu)^{\frac{1}{2}}\,\mathbf{v}_2. \tag{2.19}$$

It may be noted that in each wake the magnetic energy $\frac{1}{2}\mu h^2$ and kinetic energy $\frac{1}{2}\rho v^2$ per unit volume associated with the perturbed fields are equal. Equations (2.8) and (2.19) show that the variation of $p+\frac{1}{2}\mu H^2$ across each wake is negligible. This is the magnetohydrodynamic extension of the result that p has negligible variation across a non-conducting boundary layer or wake.

When $\alpha = 0$, so that **a** and **b** are both in the x-direction, the analysis requires modification. The last term in each curly bracket in (2.14) vanishes instead of being dominant. Equation (2.11) now factorizes to give

$$\left(\nabla^2 - k_1\frac{\partial}{\partial x}\right)\left(\nabla^2 - k_2\frac{\partial}{\partial x}\right)\boldsymbol{\omega} = 0, \tag{2.20}$$

where
$$\begin{aligned} k_{1,2} &= (U_0/2\eta\nu)\{(\eta+\nu) \pm [(\eta-\nu)^2 + 4\eta\nu\beta]^{\frac{1}{2}}\} \\ &= 2(1-\beta)\,U_0\{(\eta+\nu) \mp [(\eta-\nu)^2+4\eta\nu\beta]^{\frac{1}{2}}\}^{-1}. \end{aligned} \tag{2.21}$$

It is readily proved that the solution of (2.20) is given by $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$, where

$$\nabla^2\boldsymbol{\omega}_1 - k_1\,\partial\boldsymbol{\omega}_1/\partial x = 0, \quad \nabla^2\boldsymbol{\omega}_2 - k_2\,\partial\boldsymbol{\omega}_2/\partial x = 0, \tag{2.22}$$

and thus there are two wakes superposed, each in the stream direction. From (2.21), k_1 is always positive, but k_2 is positive or negative according as β is less than or greater than unity. A negative value of k_2 implies that the second wake is in the upstream direction. Using (2.22) and (2.10) we find that

$$\mathbf{j}_{1,2} = -(2\nu H_0/U_0)\{(\eta-\nu) \pm [(\eta-\nu)^2+4\eta\nu\beta]^{\frac{1}{2}}\}^{-1}\,\boldsymbol{\omega}_{1,2}, \tag{2.23}$$

$$\mathbf{h}_{1,2} = -(2\nu H_0/U_0)\{(\eta-\nu) \pm [(\eta-\nu)^2+4\eta\nu\beta]^{\frac{1}{2}}\}^{-1}\,\mathbf{v}_{1,2}. \tag{2.24}$$

This is a simple extension to three dimensions of Hasimoto's (1960) two-dimensional result.

It is noteworthy that, unless $\eta = \nu$, the values given by (2.21), (2.23) and (2.24), for $\alpha = 0$, are quite unrelated to the limiting values as $\alpha \to 0$ given by (2.17), (2.18) and (2.19), though both wakes are downstream for $\beta < 1$, and one is upstream and one downstream for $\beta > 1$, by each approach. The discrepancy is no surprise in view of the step from (2.14) to (2.15), but it does indicate that results for alined fields (often a convenient case to treat mathematically) should be used with caution in predicting what happens if the fields are not perfectly alined. When $\eta = \nu$, so that the magnetic and viscous diffusivities are equal, there is complete agreement between the results for $\alpha = 0$ and in the limit as $\alpha \to 0$. For alined fields with $\beta = 1$, both (2.17) and (2.21) indicate that a singular situation arises. Each gives $\nabla^2 \boldsymbol{\omega}_2 = 0$, and so $\boldsymbol{\omega}_2$ satisfies the equation for Stokes flow and not the wake equation. For this special case the whole analysis would have to be re-examined.

The detailed flow in the wakes when the Reynolds number and magnetic Reynolds number are large can be deduced immediately from the known results for a non-conducting wake at large Reynolds number. Solutions occur of two distinct types: with vortex lines in rings (not necessarily circular) about the wake axis, associated with the drag and giving a velocity defect in the wake; and with vortex lines trailing along the wake, as for a lifting wing. In the former case the volume flux defect in the wake is constant, and in the latter the strength of the equivalent vortex doublet is constant along the wake. These facts will be sufficient for our needs in this paper. They apply even if the flow in the wake is turbulent, since they are derived from over-all momentum considerations, and do not depend on the precise nature of the viscous stresses in the wake region. Furthermore, they apply whether or not the distribution of vorticity has reached its final similarity form. This is important, since magnetohydrodynamic wakes are not in general thin initially (as are the wakes behind streamlined bodies in non-conducting fluids), but start with the projected area of the body in the wake direction. In consequence, the flow perturbations may become small and the Reynolds number and the magnetic Reynolds number become large at distances far smaller than those at which the current and vorticity distributions approach their ultimate forms.

Finally it may be noted that, for $\alpha \neq 0$, fluid particles travel through the wake, since they move with a velocity differing only slightly from $\mathbf{U}_0$. Particles enter the wake, acquire current and vorticity, but then pass out of the wake, relinquish their current and vorticity, and resume their previous unruffled career. This is in sharp contrast to what happens in a non-conducting fluid, where particles remain within the wake once they have entered it.

3. Disturbances to the stream

We now turn to the flow in the main body of the fluid. This will be everywhere free from electric current or vorticity, but will have singularities at the wakes and at the body. Since we are concerned only with the flow at large distances, the effect of the body can be represented by a singularity at the origin O.

It will be helpful to review the facts about the similar problem for a non-conducting fluid. The flow outside the wake then has a term as for a source of strength Q at the origin, continuity being preserved by an inflow of strength Q along the wake, which of course lies downstream. There is also a bound vortex singularity at the origin of strength $\mathbf{K}$, perpendicular to the stream $\mathbf{U}_0$, and a trailing vortex doublet, also of strength $\mathbf{K}$, along the wake. The equivalent horse-shoe vortex $ABCD$ is illustrated in figure 2. If the circulation round the vortex is Γ and the vector BC is $\mathbf{h}$, then $\mathbf{K} = \Gamma \mathbf{h}$. As far as the flow at large distances is concerned, we may consider the limiting case $\mathbf{h} \to 0$, but $\mathbf{K}$ finite and non-zero. The lift $\mathbf{L}$ and drag $\mathbf{D}$ on the body are related to $\mathbf{K}$ and Q by the equations

$$\mathbf{D} = \rho \mathbf{U}_0 Q, \quad \mathbf{L} = \rho \mathbf{U}_0 \times \mathbf{K}. \tag{3.1}$$

Formulae equivalent to these were developed by Goldstein (1929, 1931); they are extensions of the formulae for two-dimensional flow given by Filon (1926) and Taylor (1925). The latter is itself the extension to viscous flow of the Kutta–Joukowski theorem for inviscid fluids. As far as the over-all forces are concerned there are three degrees of freedom, the three components of force being proportional to Q and the two components of $\mathbf{K}$ (which has no component parallel to $\mathbf{U}_0$).

What would we anticipate to be the equivalent situation in magnetohydrodynamics? As well as Q and $\mathbf{K}$ for each wake there are now also the corresponding magnetic quantities, the magnetic pole strength M at the origin, and the bound current strength $\mathbf{C}$ (equal to the product of the current and its length). We might therefore expect twelve degrees of freedom, three from the magnetic parameters

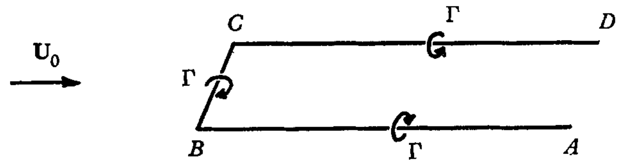

FIGURE 2. Representation of the bound and trailing vorticity in a wake.

and three from the velocity parameters in each wake. However, equations (2.18) and (2.19) reduce the number at once to six, M being related to Q and $\mathbf{C}$ to $\mathbf{K}$ in each wake. We shall now show that the equations for the main flow enable us to obtain relations which reduce the number of independent parameters from six to three, the same number as for non-conducting flow.

Outside the wakes $\mathbf{j} = \boldsymbol{\omega} = 0$ and (2.1) reduces to $\nabla(p+\frac{1}{2}\rho q^2) = 0$, or

$$p+\tfrac{1}{2}\rho q^2 = \text{constant}, \tag{3.2}$$

which is Bernoulli's equation for the pressure in its usual non-conducting form. Equation (2.4) becomes

$$(\mathbf{q}.\nabla)\mathbf{H} = (\mathbf{H}.\nabla)\mathbf{q}. \tag{3.3}$$

For small disturbances as in (2.6) this gives

$$(\mathbf{U}_0.\nabla)\mathbf{h} = (\mathbf{H}_0.\nabla)\mathbf{v},$$

and hence, since $\nabla\times\mathbf{h} = \nabla\times\mathbf{v} = 0$,

$$\nabla(\mathbf{U}_0.\mathbf{h}) = \nabla(\mathbf{H}_0.\mathbf{v}),$$

and therefore, since at infinity $\mathbf{h} = \mathbf{v} = 0$ in most directions,

$$\mathbf{U}_0.\mathbf{h} = \mathbf{H}_0.\mathbf{v}. \tag{3.4}$$

This is the fundamental equation which the magnetic and velocity perturbations must obey in the main flow.

All contributions to $\mathbf{h}$ and $\mathbf{v}$ arise directly or indirectly (as a consequence of solenoidality) from one or other of the two wakes. (It is assumed that there is no physical source or magnetic pole at the body. The extension of the theory of this paper to such cases presents no special difficulty.) We therefore write

$$\mathbf{h} = \mathbf{h}_1+\mathbf{h}_2, \quad \mathbf{v} = \mathbf{v}_1+\mathbf{v}_2, \tag{3.5}$$

where, for $\alpha \neq 0$, these quantities are related by (2.19). Equation (3.4) becomes

$$(\mathbf{a}+\beta^{\frac{1}{2}}\mathbf{b}).\mathbf{v}_1 = (\mathbf{a}-\beta^{\frac{1}{2}}\mathbf{b}).\mathbf{v}_2. \tag{3.6}$$

The first vector in each scalar product is in the direction of the appropriate wake.

Contributions to (3.6) arise from the source and the bound vortex at O. Ring vorticity in a wake produces inflow within the wake but negligible flow outside, and the velocity due to the trailing vorticity, as given by the Biot–Savart formula, is everywhere perpendicular to the wake and so does not affect the scalar product in (3.6). Due to the source and the bound vortex, the contribution $\mathbf{v}_1^*$ to $\mathbf{v}_1$ at the point with position vector $\mathbf{r}$ is given by

$$\mathbf{v}_1^* = \frac{Q_1}{4\pi r^3}\mathbf{r}+\frac{\mathbf{K}_1\times\mathbf{r}}{4\pi r^3}. \tag{3.7}$$

The last term is the Biot–Savart velocity field of the bound vortex element. (It may be remarked that $\mathbf{v}_1^*$ is not irrotational. To obtain the full irrotational field $\mathbf{v}_1$ the Biot–Savart velocity due to the trailing vorticity must also be included.) We thus have to satisfy the equation

$$(\mathbf{a}+\beta^{\frac{1}{2}}\mathbf{b}).\mathbf{v}_1^* = (\mathbf{a}-\beta^{\frac{1}{2}}\mathbf{b}).\mathbf{v}_2^*. \tag{3.8}$$

Since $\mathbf{K}_1$ is perpendicular to $\mathbf{s}_1$ we may write

$$\mathbf{K}_1 = K_{1t}\mathbf{t}_1+K_{1n}\mathbf{n}_1, \tag{3.9}$$

where the unit vectors $\mathbf{t}_1$ and $\mathbf{n}_1$ are given by (2.13). Then

$$\mathbf{s}_1.(\mathbf{K}_1\times\mathbf{r}) = (\mathbf{s}_1\times\mathbf{K}_1).\mathbf{r} = (K_{1t}\mathbf{n}_1-K_{1n}\mathbf{t}_1).\mathbf{r}.$$

Each side of (3.8) can now be expressed as a sum of multiples of $\mathbf{r}.\mathbf{a}$, $\mathbf{r}.\mathbf{b}$ and $\mathbf{r}.(\mathbf{a}\times\mathbf{b})$. The coefficients must balance, which leads to the set of equations

$$Q_1-(\cot\alpha+\beta^{\frac{1}{2}}\operatorname{cosec}\alpha)\,K_{1t} = Q_2-(\cot\alpha-\beta^{\frac{1}{2}}\operatorname{cosec}\alpha)\,K_{2t}, \tag{3.10}$$

$$\beta^{\frac{1}{2}}Q_1+(\operatorname{cosec}\alpha+\beta^{\frac{1}{2}}\cot\alpha)\,K_{1t} = -\beta^{\frac{1}{2}}Q_2+(\operatorname{cosec}\alpha-\beta^{\frac{1}{2}}\cot\alpha)\,K_{2t}, \tag{3.11}$$

$$(1+2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{\frac{1}{2}}\,K_{1n} = (1-2\beta^{\frac{1}{2}}\cos\alpha+\beta)^{\frac{1}{2}}\,K_{2n} = \tfrac{1}{2}T, \quad \text{say}. \tag{3.12}$$

These are the three extra relations which reduce the number of degrees of freedom in the flow to three. Using (2.19), equations (3.10) and (3.11) can be written as

$$M-C_t\cot\alpha+(H_0/U_0)\,K_t\operatorname{cosec}\alpha = 0,$$
$$(H_0/U_0)\,Q-C_t\operatorname{cosec}\alpha+(H_0/U_0)\,K_t\cot\alpha = 0,$$

where Q and M are the total source and pole strengths at the origin, and K_t and C_t are the t-components of the bound vortex strength and bound current strength. These may be combined into the single complex equation

$$M+iC_t = (H_0/U_0)\,e^{i\alpha}\,(Q+iK_t). \tag{3.13}$$

This is precisely the equation given by Hasimoto (1960) for his special cases of two-dimensional flow, where Q and K_t become the source strength per unit span and the circulation, and similarly for M and C_t.

An interesting corollary of this equation is that for $\alpha \neq 0$ the standard thin wing flow, in which the wake contains trailing vorticity but no ring vorticity (with its associated inflow along the wake and source flow outside) has no immediate magnetohydrodynamic analogue. Equation (3.13) shows that if $K_t \neq 0$, one at least of Q and M must be non-zero, and (2.18) now shows that both ring vorticity and ring current must be present in at least one wake.

The total bound vortex strength in the plane of $\mathbf{U}_0$ and $\mathbf{H}_0$ is

$$K_{1n}\mathbf{n}_1 + K_{2n}\mathbf{n}_2 = \mathbf{K}^*, \quad \text{say.}$$

Using (3.12) and (2.13) we obtain

$$\mathbf{K}^* = T\operatorname{cosec}\alpha(1-2\beta\cos 2\alpha+\beta^2)^{-1}\{(\beta-1)\cos\alpha\,\mathbf{a}+(1-\beta\cos 2\alpha)\,\mathbf{b}\}. \quad (3.14)$$

Similarly, the bound current strength in this plane is

$$\mathbf{C}^* = (H_0/U_0)\,T\operatorname{cosec}\alpha(1-2\beta\cos 2\alpha+\beta^2)^{-1}\{(\beta-\cos 2\alpha)\,\mathbf{a}+(1-\beta)\cos\alpha\,\mathbf{b}\}. \quad (3.15)$$

For $\alpha = 0$, (3.4) and (3.7) continue to apply, and we can deduce at once that (3.13) still holds. We may choose $\mathbf{t}$ so that $\mathbf{K}$ (and therefore also $\mathbf{C}$) lies in the t-direction, so there is no need to introduce $\mathbf{K}^*$ and $\mathbf{C}^*$ in this case.

4. Forces on the body

In steady flow the momentum inside any large fixed surface S enclosing the body remains constant. Consequently the flux of momentum out of S must be balanced by the total force exerted across S by the fluid outside due to fluid pressure and the Maxwell stresses, plus the reaction of the body on the fluid. Suppose that the force on the body is $\mathbf{F}$, so that the reaction on the fluid is $-\mathbf{F}$. Momentum conservation requires that

$$\mathbf{F} = -\int_S \{\rho\mathbf{q}q_n + p\mathbf{n} + \tfrac{1}{2}\mu H^2\mathbf{n} - \mu\mathbf{H}H_n\}\,dS. \quad (4.1)$$

The terms in the integrand represent respectively the effects of the momentum flux, the pressure p, the magnetic pressure $\frac{1}{2}\mu H^2$, and the Maxwell tension μH^2 along the magnetic lines of force. The unit outward normal to dS is $\mathbf{n}$, and the suffix n denotes the component of a vector in the direction of $\mathbf{n}$. As in non-conducting flow, viscous forces (which are zero outside the wakes) make a negligible contribution to the surface integral.

We now write $\mathbf{q} = \mathbf{U}_0 + \mathbf{v}$, $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$, as in (2.6), and ignore squares of $\mathbf{v}$ and $\mathbf{h}$. We also use the relations

$$\int_S \mathbf{n}\,dS = \int_S q_n\,dS = \int_S H_n\,dS = 0,$$

the last two being consequences of (2.3). We again assume that there is no physical source or magnetic pole at the body. Equation (4.1) becomes

$$\mathbf{F} = -\int_S \{\rho\mathbf{v}U_{0n} + (p-p_0)\,\mathbf{n} + \mu\mathbf{h}\,.\,\mathbf{H}_0\mathbf{n} - \mu\mathbf{h}H_{0n}\}\,dS, \quad (4.2)$$

where p_0 is the pressure at infinity. Outside the wakes we have Bernoulli's equation (3.2), which becomes

$$p - p_0 = -\rho \mathbf{v} . \mathbf{U}_0, \tag{4.3}$$

ignoring v^2. Suppose that

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_w, \tag{4.4}$$

where $\mathbf{F}_0$ is the value of $\mathbf{F}$ obtained by using (4.3), and $\mathbf{F}_w$ is the extra contribution due to (4.3) not being applicable in the wakes. Then from (4.2),

$$\mathbf{F}_0 = \rho \mathbf{U}_0 \times \int_S (\mathbf{n} \times \mathbf{v})\, dS - \mu \mathbf{H}_0 \times \int_S (\mathbf{n} \times \mathbf{h})\, dS. \tag{4.5}$$

We may further split up the problem by writing

$$\mathbf{F}_0 = \mathbf{F}_S + \mathbf{F}_B + \mathbf{F}_T, \tag{4.6}$$

where $\mathbf{F}_S$ is due to the source (and pole) at the origin, $\mathbf{F}_B$ is due to the bound vortex (and current) at the origin, and $\mathbf{F}_T$ is due to the trailing vortices (and currents). As remarked before, ring vorticity and current produce no effect in the main flow.

If we take S as a sphere centre O, we see at once from (4.5) that $\mathbf{F}_S = 0$, since the corresponding $\mathbf{v}$ and $\mathbf{h}$ are parallel to $\mathbf{n}$, and so the vector products in the integrands are zero.

For $\mathbf{F}_T$ consider each wake separately. For the $\mathbf{s}_1$ wake, take S as the planes $s_1 = \pm A$. For each trailing element of the equivalent horse-shoe vortex (as in figure 2), $\mathbf{n} \times \mathbf{v}_1$ is in the plane of S, and if we sum over circular rings of S, centre the point at which the trailing element cuts S, we see by symmetry that the total contribution is zero. The same is true for $\mathbf{h}_1$, and also for the $\mathbf{s}_2$ wake, on taking S as $s_2 = \pm A$. Thus $\mathbf{F}_T = 0$.

For $\mathbf{F}_B$ the relevant velocity field is, from (3.7), $\mathbf{v} = (\mathbf{K} \times \mathbf{r})/4\pi r^3$. In order not to be led into error through using the non-irrotational Biot–Savart field, we must be careful to use the same surfaces S as were used in discussing $\mathbf{F}_T$. Accordingly we write $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$ and consider the two parts separately. For $\mathbf{K}_1$, we take S as the surfaces $s_1 = \pm A$, as before. If $\mathbf{v}_1$ is the corresponding velocity,

$$\mathbf{n} \times \mathbf{v}_1 = A\mathbf{K}_1/4\pi r^3$$

on each of $s_1 = \pm A$, since $\mathbf{n}.\mathbf{K}_1 = 0$. For the first integral in (4.5) we write $r = (A^2 + u^2)^{\frac{1}{2}}$, $dS = 2\pi u\, du$, and obtain as the contribution to $\mathbf{F}_B$

$$2\rho \mathbf{U}_0 \times \mathbf{K}_1 \int_0^\infty \tfrac{1}{2} A u (A^2 + u^2)^{-\frac{3}{2}}\, du = \rho \mathbf{U}_0 \times \mathbf{K}_1.$$

The initial factor 2 arises since there are equal contributions from the two surfaces $s_1 = \pm A$. Similar results hold for $\mathbf{K}_2$ and for the bound currents $\mathbf{C}_1$ and $\mathbf{C}_2$. Combining all these results, we have from (4.6)

$$\mathbf{F}_0 = \mathbf{F}_B = \rho \mathbf{U}_0 \times \mathbf{K} - \mu \mathbf{H}_0 \times \mathbf{C}. \tag{4.7}$$

In the wakes (4.3) no longer applies, but on the other hand, as shown in § 2, $p + \frac{1}{2}\mu H^2$ is constant across the wake. Consequently the contributions to $\mathbf{F}_w$ from the second and third terms in the integrand in (4.1) or (4.2) cancel. Thus

$$\mathbf{F}_w = -\rho U_{0n} \int_{S_w} \mathbf{v}\, dS + \mu H_{0n} \int_{S_w} \mathbf{h}\, dS, \tag{4.8}$$

where S_w is the intersection of S with the wakes, and $\mathbf{v}$ and $\mathbf{h}$ now denote the longitudinal velocity and magnetic fields due to ring vorticity and current. Again we consider each wake separately. For convenience, take S so that S_w cuts the wakes at right angles. In the $\mathbf{s}_1$ wake, for $\alpha \neq 0$, $\mathbf{h}_1 = -(\rho/\mu)^{\frac{1}{2}}\mathbf{v}_1$ from (2.19) and

$$\int_{S_w} \mathbf{v}_1 dS = -Q_1 \mathbf{s}_1.$$

(The negative sign arises since a source in the main flow is accompanied by an equal inflow along the wake.) Consequently

$$\begin{aligned}\mathbf{F}_{w1} = \rho\{U_{0n} + (\mu/\rho)^{\frac{1}{2}} H_{0n}\} Q_1 \mathbf{s}_1 &= \rho\{\mathbf{U}_0 + (\mu/\rho)^{\frac{1}{2}} \mathbf{H}_0\} Q_1 \\ &= \rho \mathbf{U}_0 Q_1 - \mu \mathbf{H}_0 M_1,\end{aligned} \qquad (4.9)$$

using (2.19). Similarly, $\mathbf{F}_{w2} = \rho\mathbf{U}_0 Q_2 - \mu\mathbf{H}_0 M_2$, and adding these we obtain

$$\mathbf{F}_w = \rho\mathbf{U}_0 Q - \mu\mathbf{H}_0 M. \qquad (4.10)$$

When $\alpha = 0$, (4.10) is an immediate consequence of (4.8).

Combining these results, we have from (4.4)

$$\mathbf{F} = \rho\mathbf{U}_0 \times \mathbf{K} - \mu\mathbf{H}_0 \times \mathbf{C} + \rho\mathbf{U}_0 Q - \mu\mathbf{H}_0 M. \qquad (4.11)$$

It will be observed that the classical results (3.1) are recovered as a special case by putting $\mathbf{H}_0 = 0$.

The contributions to (4.11) due to Q, M, K_t and C_t are all in the plane of $\mathbf{U}_0$ and $\mathbf{H}_0$, while those due to $\mathbf{K}^*$ and $\mathbf{C}^*$ are perpendicular to this plane. Suppose that

$$\mathbf{F} = (X, Y, Z),$$

where the components X, Y, Z are in the directions of $\mathbf{a}$, $\mathbf{t}$, $\mathbf{a}\times\mathbf{t}$ respectively. (Thus Z is the component perpendicular to $\mathbf{U}_0$ in the plane of $\mathbf{U}_0$ and $\mathbf{H}_0$.) Using (3.13), (4.11) gives

$$\begin{aligned}X + iZ &= \rho U_0(Q + iK_t) - \mu H_0 e^{i\alpha}(M + iC_t) \\ &= \rho U_0(1 - \beta e^{2i\alpha})(Q + iK_t).\end{aligned} \qquad (4.12)$$

This again is identical in form with the formula derived by Hasimoto (1960) for his particular cases of two-dimensional flow. Also, using (3.14) and (3.15) in (4.11),

$$\begin{aligned}Y &= \rho U_0 T(1 - 2\beta\cos 2\alpha + \beta^2)^{-1}\{-(1 - \beta\cos 2\alpha) - \beta(\beta - \cos 2\alpha)\} \\ &= -\rho U_0 T.\end{aligned} \qquad (4.13)$$

The expressions (4.12) and (4.13) for the force on the body are remarkably compact. Taken in conjunction with (3.13), (3.14) and (3.15), they describe the over-all effects on the body and on the velocity and magnetic fields in terms of three independent parameters, the same as in the non-conducting case.

5. The electric field

The electric field $\mathbf{E}$ was eliminated as early as equation (2.4) and has not had to be reintroduced. Now that the description of the velocity and magnetic fields has been completed, $\mathbf{E}$ can be obtained at once. In the undisturbed flow, (2.2) shows that there is a uniform electric field

$$\mathbf{E}_0 = -\mu\mathbf{U}_0 \times \mathbf{H}_0 = \mu U_0 H_0 \sin\alpha\, \mathbf{t}. \qquad (5.1)$$

Taking the divergence of (2.2) and neglecting squares and products of perturbations, we obtain

$$\nabla . \mathbf{E} = -\mu \nabla . (\mathbf{q} \times \mathbf{H}) = \mu(\mathbf{U}_0 . \mathbf{j} - \mathbf{H}_0 . \boldsymbol{\omega}). \tag{5.2}$$

The equation for the charge density ρ_e is

$$\epsilon \nabla . \mathbf{E} = \rho_e, \tag{5.3}$$

where ϵ is the permittivity, and so (5.2) gives the electric charge distribution which modifies the basic field $\mathbf{E}_0$.

Contributions to (5.2) arise only from terms associated with the two wakes. For $\alpha \neq 0$, (2.18) gives for the $\mathbf{s}_1$ wake

$$\begin{aligned} \nabla . \mathbf{E}_1 &= \mu\{\mathbf{U}_0 + (\mu/\rho)^{\frac{1}{2}} \mathbf{H}_0\} . \mathbf{j}_1 \\ &= \mu U_0 (1 + 2\beta^{\frac{1}{2}} \cos \alpha + \beta)^{\frac{1}{2}} \mathbf{s}_1 . \mathbf{j}_1. \end{aligned} \tag{5.4}$$

The ring current in the wake and the bound current at the origin are each perpendicular to $\mathbf{s}_1$, so only the trailing current contributes to (5.4). From (5.4) and (5.3), the trailing current doublet $\mathbf{C}_1$ gives rise to an electric line doublet of strength

$$\mathbf{D}_1 = \epsilon \mu U_0 (1 + 2\beta^{\frac{1}{2}} \cos \alpha + \beta)^{\frac{1}{2}} \mathbf{C}_1. \tag{5.5}$$

Similarly, along the $\mathbf{s}_2$ wake there is an electric line doublet

$$\mathbf{D}_2 = \epsilon \mu U_0 (1 - 2\beta^{\frac{1}{2}} \cos \alpha + \beta)^{\frac{1}{2}} \mathbf{C}_2. \tag{5.6}$$

When $\alpha = 0$ it is again clear that only trailing current and vorticity affect (5.2). From (2.23) the strength $\mathbf{D}_1$, $\mathbf{D}_2$ of the electric doublets in the twp wakes are found to be

$$\mathbf{D}_{1,2} = (\epsilon \mu U_0 / 2\nu) \{(\eta + \nu) \pm [(\eta - \nu)^2 + 4\eta\nu\beta]^{\frac{1}{2}}\} \mathbf{C}_{1,2}. \tag{5.7}$$

As in § 2, there is no agreement between the values for $\alpha = 0$ given by (5.7), and the limiting values as $\alpha \to 0$ given by (5.5) and (5.6), except when $\eta = \nu$. But in all cases the total electric field at large distances from the body is seen to consist of the uniform field $\mathbf{E}_0$, together with the fields due to semi-infinite line doublets along the two wakes.

REFERENCES

Filon, L. N. G. 1926 *Proc. Roy. Soc.* A, **113**, 7.

Goldstein, S. 1929 *Proc. Roy. Soc.* A, **123**, 216.

Goldstein, S. 1931 *Proc. Roy. Soc.* A, **131**, 198.

Hasimoto, H. 1960 *Rev. Mod. Phys.* **32**, 860.

Lary, E. C. 1962 *J. Fluid Mech.* **12**, 209.

McCune, J. E. 1960 *J. Fluid Mech.* **7**, 449.

Sears, W. R. & Resler, E. L. 1959 *J. Fluid Mech.* **5**, 257.

Stewartson, K. 1960 *J. Fluid Mech.* **8**, 82.

Stewartson, K. 1961 *Z. Angew. Math. Phys.* **12**, 261.

Taylor, G. I. 1925 *Phil. Trans.* A, **225**, 238.